Emad A. Zikry, President and Chief Executive Officer

Calculating Implied Default Rates from CDS Spreads

Introduction

Credit market investors have to assess yield against the probability of default constantly. In this regard, many tools have been developed to help investors to estimate the default probabilities. Rating agencies publish actuarial tables regularly based on the long history of default experiences. Other tools such as CreditSights' BondScore or KMV's EDF combine company financials and equity market valuations with historical default experiences to estimate default probabilities. It is important to realize that when market provides a CDS quote on a credit, market is actually providing a market-implied default probability, which differ from experience-based estimates in a fundamental way. In turbulent times fear causes markets to over-estimate default probabilities, and in good times greed causes market to under-estimate default probabilities.

In the following we shall discuss the mathematics involved to calculate market implied default probabilities from term structures of CDS curves. The market implied default probabilities calculated from the algorithm outlined below have wide applications in credit trading and risk management, ranging from calculating the present value of a CDS contract to the fair value spreads of various synthetic CDS tranches.

Notations

$B(t)$: Risk-free discount factor at time t.
$Q(t)$: The survival probability from time period $(0,t)$
c : CDS spreads
t_i : CDS payment dates. $i = 1,2,...N$. $t_0 = 0$ is the starting date, $t_N = T$ is the ending date.
R : Recovery rate

Basic Equations

The basic idea in determining the fair value of CDS spreads is that the expected present value of CDS premium payments must be equal to the expected present value of payout in the event of default. The expected present value of CDS premium payments is

$$c\sum_{i=1}^{N}(t_i - t_{i-1})B(t_i)Q(t_i) - c\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(t - t_{i-1})B(t)dQ(t)$$

The 2[nd] term represents the amount of payment accrued in the event that default occurs at time t between t_{i-1} and t_i (remember that $dQ(t) < 0$). When day counting conventions are considered, the CDS spread should be adjusted by the appropriate factors. This expected present value of payments should equal to the expected present value of the payout from the CDS contract in case of default:

$$(1-R)\int_0^T B(t)dQ(t)$$

Therefore, we have

$$c\sum_{i=1}^{N}(t_i - t_{i-1})B(t_i)Q(t_i) - c\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(t - t_{i-1})B(t)dQ(t) + (1-R)\int_0^T B(t)dQ(t) = 0 \qquad (1)$$

Constant Hazard Rates

We apply equation (1) to the whole term structure of CDS curve: $\{c_j\}$ with maturity dates $\{T_j\}$. Furthermore, we assume that hazard rate over time period (T_{j-1}, T_j) is constant h_j.

Under the piece-wise constant hazard rate approximation we then have

$$Q(t) = Q(T_{j-1})\exp(-h_j(t - T_{j-1})),\ T_{j-1} \le t \le T_j \qquad (2)$$

In order to proceed further, we introduce a new set of notations. Over the time period (T_{j-1}, T_j), we denote the payment dates as $\{t_m^j\}$, where $m = 0,1,2,....n_j$, with $t_0^j = T_{j-1}$ and $t_{n_j}^j = T_j$. The CDS premium covering the time $(0, T_j)$ is c_j. Equation (1) can then be re-written as

$$c_k\sum_{j=1}^{k}\sum_{m=1}^{n_j}(t_m^j - t_{m-1}^j)B(t_m^j)Q(t_m^j) - c_k\sum_{j=1}^{k}\sum_{m=1}^{n_j}\int_{t_{m-1}^j}^{t_m^j}(t - t_{m-1}^j)B(t)dQ(t) + (1-R)\sum_{j=1}^{k}\sum_{m=1}^{n_j}\int_{t_{m-1}^j}^{t_m^j}B(t)dQ(t) = 0$$

Bootstrapping

We also apply exponential extrapolation to the discount curve:

$$B(t) = B(t_{m-1}^j)\exp(-r_m^j(t - t_{m-1}^j)), \qquad t_{m-1}^j \le t \le t_m^j \tag{3}$$

Where

$$r_m^j = -\frac{\log(B(t_m^j)/B(t_{m-1}^j))}{t_m^j - t_{m-1}^j} \tag{4}$$

Combining Equations (1), (2) and (3), we have

$$\begin{aligned}
&c_k \sum_{j=1}^{k}\sum_{m=1}^{n_j}(t_m^j - t_{m-1}^j)B(t_m^j)Q(T_{j-1})\exp(-h_j(t_m^j - T_{j-1})) + \\
&c_k \sum_{j=1}^{k}\sum_{m=1}^{n_j}B(t_{m-1}^j)Q(T_{j-1})h_j \int_{t_{m-1}^j}^{t_m^j}(t - t_{m-1}^j)\exp(-r_m^j(t - t_{m-1}^j))\exp(-h_j(t - T_{j-1}))dt - \\
&(1-R)\sum_{j=1}^{k}\sum_{m=1}^{n_j}B(t_{m-1}^j)Q(T_{j-1})h_j \int_{t_{m-1}^j}^{t_m^j}\exp(-r_m^j(t - t_{m-1}^j))\exp(-h_j(t - T_{j-1}))dt = 0
\end{aligned}$$

After a change of variable $\tau = t - t_{m-1}^j$ and let $\Delta t_m^j = t_m^j - t_{m-1}^j$, we have

$$\begin{aligned}
&c_k \sum_{j=1}^{k}\sum_{m=1}^{n_j}(t_m^j - t_{m-1}^j)B(t_m^j)Q(T_{j-1})\exp(-h_j(t_m^j - T_{j-1})) + \\
&c_k \sum_{j=1}^{k}\sum_{m=1}^{n_j}B(t_{m-1}^j)Q(T_{j-1})h_j \exp(-h_j(t_{m-1}^j - T_{j-1}))\int_{t_{m-1}^j}^{t_m^j}\tau\exp(-(r_m^j + h_j)\tau)d\tau - \\
&(1-R)\sum_{j=1}^{k}\sum_{m=1}^{n_j}B(t_{m-1}^j)Q(T_{j-1})h_j \exp(-h_j(t_{m-1}^j - T_{j-1}))\int_{t_{m-1}^j}^{t_m^j}\exp(-(r_m^j + h_j)\tau)d\tau = 0
\end{aligned} \tag{5}$$

The two integrals in Equation (5) can be carried out explicitly

$$I(\gamma,\Delta) = \int_0^{\Delta}\exp(-\gamma\tau)d\tau = \frac{1-\exp(-\gamma\Delta)}{\gamma} \tag{6}$$

$$J(\gamma,\Delta) = \int_0^{\Delta}\tau\exp(-\gamma\tau)d\tau = -\frac{\partial I(\gamma,\Delta)}{\partial\gamma} = \frac{1-(1+\gamma\Delta)\exp(-\gamma\Delta)}{\gamma^2} \tag{7}$$

Equation (5) can be solved numerically along the credit curve, in much the same way as a risk free discount curve is constructed from a set of government bonds.

Limit of Instantaneous CDS

CDS uses actual/360 day-counting convention. This can easily be adjusted by multiplying a factor of 365/360 to $\{c_k\}$ in the above equations. In the limit of instantaneous CDS contract, Equation (1) reduces to

$$c\Delta t = (1-R)\Delta P$$

This gives the instantaneous hazard rate of

$$H(0) = \frac{\Delta P}{\Delta t} = \frac{c}{1-R} \tag{8}$$

Limit of Infinitesimal CDS premiums

In the limit of small CDS premiums, the second term in Equation (1) is of order O (c^2) and can be dropped. Equation (1) reduces to

$$c\sum_{i=1}^{N}(t_i - t_{i-1})B(t_i) - (1-R)H\int_0^T B(t)dt = 0$$

This gives the relationship between c and H

$$c = \frac{(1-R)H\int_0^T B(t)dt}{\sum_{i=1}^{N}(t_i - t_{i-1})B(t_i)} \tag{9}$$

Equations (8) and (9) are useful in checking numerical algorithms.

Vanderbilt Research Team

Emad A. Zikry

Chief Executive Officer
Vanderbilt Avenue Asset Management

Emad is the Managing Partner and Chief Executive Officer of Vanderbilt Avenue Asset Management LLC. Vanderbilt's client base includes Multi-national Corporations, Public Funds, Foundations/Endowments, and Taft Hartley accounts.

Previously, Emad was Chairman of Institutional Business at Pioneer Investments. Pioneer investments has more than $300 Billion in assets under management. The parent of Pioneer, UniCredit S.p.A., is the largest bank in Italy and the second largest bank in Europe. Pioneer had purchased Vanderbilt Capital Advisors, of which Emad was the founder and Chief Executive Officer.

Emad has had numerous articles published in professional and academic journals such as *The Journal of Forecasting*, *The American Economist* and *The Journal of Fixed Income*. He is a Board member of The National Investment Company. Emad was a member of the Board of Advisors of the Pacific Institute, The Advisory Committee of Fulcrum Global Partners, The Chief Executive Officers Club and formerly a board member of The Foreign Policy Association. He also served on the Board of Directors of the University of Albany Foundation, NextGen Healthcare Inc., The Park Avenue Bank, AA Bank and The New Providence Fund and Associates LP.

Emad is an FINRA Arbitrator. He is also a member of the National Association for Business Economists and The Economic Club of New York. Emad served as an adjunct professor at the University of Kansas and St. John's University.

Emad holds a Bachelor of Science from the University of Albany, and a M.A. and Ph.D. in Economics from the University of Kansas.